Exact differential equation

28.8.18

Consider the differential equation $Mdx + Ndy = 0$, where M and N are constant or function of x, y.

$Mdx + Ndy = 0$ —— (i).

The equation (i) is said to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

1) Solve the differential equation $(x^3 - 3x^2y + 2xy^2)dx + (-x^3 + 2x^2y - y^3)dy = 0$

⇒ Here, $M = x^3 - 3x^2y + 2xy^2$ and $N = -x^3 + 2x^2y - y^3$

$\therefore \frac{\partial M}{\partial y} = -3x^2 + 4xy$ and $\frac{\partial N}{\partial x} = -3x^2 + 4xy$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\therefore$ The given differential equation is exact.

$\therefore$ i) $\int M\,\partial x = \int (x^3 - 3x^2y + 2xy^2)\,\partial x = \frac{x^4}{4} - x^3y + x^2y^2$

ii) $\int N\,\partial y = \int (-x^3 + 2x^2y - y^3)\,\partial y = -x^3y + x^2y^2 - \frac{y^4}{4}$

$\therefore$ The general solution is $\frac{x^4}{4} - x^3y + x^2y^2 - \frac{y^4}{4} = c$.

Linear differential equation (1st order) :-

The differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are constants or functions of 'x' is known as first order linear differential equation.

The integrating factor (I.F.) of the equation is $e^{\int P dx}$.

2) Solve the differential equation $\frac{dy}{dx} - \frac{3}{x+2}y = (x+2)^3$

⇒ The given equation, $\frac{dy}{dx} - \frac{3}{x+2}y = (x+2)^3$ —— (i)

The equation (i) is linear in 'y'.

The $Q = (x+2)^3$

where $P = -\frac{3}{x+2}$ and

$\therefore I.F. = e^{\int P dx} = e^{\int -\frac{3dx}{x+2}} = e^{-3\log(x+2)} = \frac{1}{(x+2)^3}$,

Multiplying both sides of (i) by I.F. we have,

$\frac{1}{(x+2)^3}\frac{dy}{dx} - \frac{3}{(x+2)^4}y = 1$

or, $\frac{d}{dx}\left(y \cdot \frac{1}{(x+2)^3}\right) = 1$.

Integrating,

$y \cdot \frac{1}{(x+2)^3} = x + c$

$\therefore$ The general solution $\frac{y}{(x+2)^3} = x + c$, (Ans).

3) Solve the differential equation $\frac{dy}{dx} + y\cot x = 2\cos x$

$\Rightarrow$ The given equation, $\frac{dy}{dx} + y\cot x = 2\cos x$ — (i) [Linear in 'y']

Here, $P = \cot x$ and $Q = 2\cos x$

$I.F. = e^{\int P dx} = e^{\int \cot x \, dx} = e^{\log|\sin x|} = \sin x$.

Multiplying both sides of (i) by I.F. and Integrating we have,

$y \cdot \sin x = \int 2\cos x \sin x \, dx + c = -\frac{\cos 2x}{2} + c$.

$\therefore$ The general solution $y \sin x = -\frac{\cos 2x}{2} + c$.

□ Linear in x :- The differential equation $\frac{dx}{dy} + Px = Q$, where P, Q are constants or functions of y, is known as first order linear differential equation (linear in 'x').

In this case, $I.F. = e^{\int P dy}$

4) Solve the differential equation $(x+y+1)\,dy = dx$

$\Rightarrow$ The equation can be written as $\frac{dx}{dy} = x + y + 1$

$\Rightarrow \frac{dx}{dy} - x = y + 1$ — (i) [Linear in x]

where, $P = -1$ and $Q = (y+1)$

$\therefore I.F. = e^{\int P dy} = e^{\int -dy} = e^{-y}$

Multiplying both sides of (i) by I.F. and Integrating we have,

$x \cdot e^{-y} = \int (y+1)e^{-y}\, dy + c$

~~$-\left(\frac{y^2}{2} + y\right)e^{-y} + \int e^{-y}\left(\frac{y^2}{2} + y\right) dy$~~

$= -(y+1)e^{-y} + \int e^{-y}\, dy = -(y+1)e^{-y} - e^{-y} + c$

$= (-y-1-1)e^{-y} = -e^{-y}(y+2) + c$

Bernoullis equation :- The differential equation of the form $\frac{dy}{dx} + Py = Qy^n$ where P, Q are constants or functions of x is known as Bernoullis equation.

5) Solve the differential equation $\frac{dy}{dx} + \frac{1}{x}y = x^2 y^6$.

=> The given equation is $\frac{dy}{dx} + \frac{1}{x}y = x^2 y^6$ —— (i)

which is Bernoullis equation.

dividing both sides of (i) by y^{6} we have,

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y^5} = x^2 \text{ —— (ii)}$$

Let, $z = \frac{1}{y^5}$

$\therefore \frac{dz}{dx} = -\frac{5}{y^6}\frac{dy}{dx}$

$\therefore \frac{1}{y^6}\frac{dy}{dx} = -\frac{1}{5}\frac{dz}{dx}$

$\therefore$ from (ii),

$$-\frac{1}{5}\frac{dz}{dx} + \frac{1}{x}z = x^2$$

or, $\frac{dz}{dx} - \frac{5}{x}z = -5x^2$ —— (iii) [Linear in z]

$\therefore$ I.F. $= e^{\int P dx} = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = \frac{1}{x^5}$

$\therefore$ Multiplying ~~(ii)~~ both sides of (iii) by I.F. and Integrating we have,

$$z\cdot\frac{1}{x^5} = \int\left(-5x^2\cdot\frac{1}{x^5}\right)dx + c = -5\int\frac{1}{x^3}dx + c$$

$$= -5\int x^{-3}dx + c = -5\frac{x^{-2}}{-2} + c$$

$$= \frac{5}{2x^2} + c$$

or, $\frac{1}{y^5 x^5} = \frac{5}{2x^2} + c$.

6) Solve the differential equation $x\frac{dy}{dx} + y = y^2\log x$.

=> The given equation can be written as,

$$\frac{dy}{dx} + \frac{y}{x} = \frac{\log x}{x}\cdot y^2 \text{ —— (i)}$$, which is Bernoulli equation.

or dividing both sides of (i) by y^2 we have,

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = \frac{\log x}{x} \text{ —— (ii)}$$

Let, $z = \frac{1}{y}$

$\therefore \frac{dz}{dy} = -\frac{1}{y^2} \cdot \frac{dy}{dx}$

or, $\frac{1}{y^2} \frac{dy}{dx} = -\frac{dz}{dy}$

$\therefore$ from (ii),

$$-\frac{dz}{dy} + \frac{1}{x} z = \frac{\log x}{x}$$

or, $\frac{dz}{dy} - \frac{1}{x} z = -\frac{\log x}{x}$ — (iii) [Linear in z]

$$\therefore I.F. = e^{\int P dx} = e^{-\int \frac{dx}{x}} = e^{-\log x} = \frac{1}{x}$$

$\therefore$ Multiplying both sides of (iii) by I.F. and Integrating we have,

$$z \cdot \frac{1}{x} = \int -\frac{\log x}{x^2} dx + c$$

$$= -\int \log x \cdot x^{-2} dx + c$$

$$= -\left[\log x \frac{x^{-1}}{-1} + \int \frac{x^{-1}}{x} dx\right] + c$$

$$= \frac{\log x}{x} - \int \frac{dx}{x^2} + c$$

$$= \frac{\log x}{x} - \frac{x^{-1}}{-1} + c$$

$$= \frac{\log x}{x} + \frac{1}{x} + c$$

$$\therefore \frac{1}{y} \cdot \frac{1}{x} = \frac{\log x + 1}{x} + c$$

$\therefore$ The general solution, $\frac{1}{y} \cdot \frac{1}{x} = \frac{1 + \log x}{x} + c$.

7) Solve the differential equation $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$

$\Rightarrow$ The given equation,

$$\sec^2 y \frac{dy}{dx} + 2x\tan y = x^3 \quad \text{(i)}$$

Let,

$z = \tan y$

$\therefore \frac{dz}{dx} = \sec^2 y \frac{dy}{dx}$

$\therefore$ from (i),

$$\frac{dz}{dx} + 2xz = x^3 \quad \text{(ii)} \quad [\text{Linear in 'z'}]$$

$\therefore$ I.F. $= e^{\int 2x\,dx} = e^{x^2}$

multiplying both sides of (ii) by I.F. and integrating we have,

Let,
$x^2 = u$
$2x\,dx = du$
$x\,dx = \frac{du}{2}$

$$z \cdot e^{x^2} = \int x^3 \cdot e^{x^2}\,dx + c$$

$$= \frac{1}{2}\int u e^u\,du + c$$

$$= \frac{1}{2}ue^u - \frac{1}{2}\int e^u\,du + c = \frac{1}{2}ue^u - \frac{1}{2}e^u + c$$

or, $\tan y \cdot e^{x^2} = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + c$

8) Solve the differential equation $(x^2y^3 + 2xy)dy = dx$

$\Rightarrow$ The given equation can be written as,

$$\frac{dx}{dy} = x^2y^3 + 2xy$$

or, $\frac{dx}{dy} - 2y \cdot x = y^3 x^2$ — (i) [Bernoulli's form]

dividing both sides (i) by x^2

$$\therefore \frac{1}{x^2}\frac{dx}{dy} - 2y \cdot \frac{1}{x} = y^3 \quad \text{(ii)}$$

Let, $z = \frac{1}{x}$

$$\therefore \frac{dz}{dy} = -\frac{1}{x^2}\frac{dx}{dy}$$

from (ii),

$$-\frac{dz}{dy} - 2yz = y^3$$

or, $\frac{dz}{dy} + 2yz = -y^3$ — (iii) [Linear in 'z']

$\therefore I.F. = e^{2\int y\,dy} = e^{y^2}$

multiplying both sides of (iii) by I.F. and integrating we have,

$z e^{y^2} = -\int y^3 e^{y^2} dy + c$

Let, $y^2 = u$, $2y\,dy = du$, $y\,dy = \frac{du}{2}$

$= -\frac{1}{2}\int u e^u du + c$

$= -\frac{1}{2}ue^u + \frac{1}{2}e^u + c$

or, $\frac{1}{x}\cdot e^{y^2} = -\frac{1}{2}y^2e^{y^2} + \frac{1}{2}e^{y^2} + c$.

9) Solve the differential equation $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$

$\Rightarrow$ The equation can be written as,

$\frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{\log y} = \frac{1}{x^2}$ —(i)

Let, $z = \frac{1}{\log y}$

$\therefore \frac{dz}{dx} = -\frac{1}{y(\log y)^2}\frac{dy}{dx}$

$\therefore \frac{1}{y(\log y)^2}\frac{dy}{dx} = -\frac{dz}{dx}$

$\therefore$ from (i),

$-\frac{dz}{dx} + \frac{1}{x}\cdot z = \frac{1}{x^2}$

or, $\frac{dz}{dx} - \frac{1}{x}\cdot z = -\frac{1}{x^2}$ —(ii) [Linear in 'z']

$\therefore I.F. = e^{-\int\frac{dx}{x}} = e^{-\log x} = \frac{1}{x}$

$\therefore$ multiplying both sides of (ii) by I.F. and integrating we have,

$z\frac{1}{x} = -\int\frac{1}{x^3}dx + c$

$= -\int x^{-3}dx + c$

or, $\frac{1}{x\log y} = -\frac{x^{-2}}{-2} + c = \frac{1}{2x^2} + c$

Higher order linear diff. equ. with constant coefficient

The equation of the form $P_0\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\dots+P_{n-1}\frac{dy}{dx}+P_ny=0$ — (i)

, where $P_0, P_1, P_2, \dots, P_n$ are constants is called higher order linear diff. equation with constant coefficient.

The equation can be written as,

$$\left(P_0D^n+P_1D^{n-1}+P_2D^{n-2}+\dots+P_{n-1}D+P_n\right)y=0, \quad D\equiv\frac{d}{dx}$$

The Auxilliary equation of the diff. equation is

$$P_0m^n+P_1m^{n-1}+P_2m^{n-2}+\dots+P_{n-1}m+P_n=0 \quad \text{— (ii)}$$

Let, $m_1, m_2, m_3, \dots, m_n$ be the roots of the Equation (ii), this root are called auxilliary root of the equation (i)

If $m_1, m_2, \dots, m_n$ are distinct then the solution of (i) is

$$y=c_1e^{m_1x}+c_2e^{m_2x}+\dots+c_ne^{m_nx}.$$

1) Solve the diff. equation $2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0$

$\Rightarrow$ The equation can be written as

$$(2D^2-3D+1)y=0 \quad \text{— (i)}$$

The auxilliary equation is

$$2m^2-3m+1=0$$
$$2m^2-2m-m+1=0$$
$$\Rightarrow 2m^2-2m-1(m-1)=0$$
$$\Rightarrow 2m(m-1)-1(m-1)=0$$
$$\Rightarrow (2m-1)(m-1)=0$$
$$\therefore m=\frac{1}{2}, 1$$

$\therefore$ The solution is $y=c_1e^{x}+c_2e^{\frac{1}{2}x}$ (Ans)

2) Solve the diff. equation $\frac{d^4y}{dx^4}-y=0$

$\Rightarrow$ The equation can be written as,

$$(D^4-1)y=0$$

$\therefore$ The auxilliary equation,

$$m^4-1=0$$
$$\text{or, } (m^2+1)(m^2-1)=0$$
$$\Rightarrow m=-1, 1, i, -i$$

$\therefore$ The general solution, $y=c_1e^{x}+c_2e^{-x}+e^{0\cdot x}\{c_3\cos x+c_4\sin x\}$

$$=c_1e^{x}+c_2e^{-x}+c_3\cos x+c_4\sin x$$

3) Solve the diff. equation $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$

$\Rightarrow$ The equation can be written as,

$$(D^4 - 4D^3 + 8D^2 - 8D + 4)y = 0 \quad \text{---(i)}$$

$\therefore$ The auxilliary equation

$$m^4 - 4m^3 + 8m^2 - 8m + 4 = 0$$

$$\Rightarrow (m^2 - 2m + 2)^2 = 0$$

$$\Rightarrow (m^2 - 2m + 2)^2 = 0$$

$$\Rightarrow m^2 - 2m + 2 = 0 \qquad , m^2 - 2m + 2 = 0$$

$$m = \frac{2 \pm \sqrt{4-8}}{2} \qquad m = 1 \pm i$$

$$= 1 \pm i$$

$\therefore$ The general solution, $y = e^x\{(c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x\}$

4) Solve the diff. equation $\frac{d^2y}{dx^2} + y = 0$, given $y = 2$ for $x = 0$ and $y = -2$ for $x = \frac{\pi}{2}$.

$\Rightarrow$ The equation can be written as, $(D^2+1)y = 0$ ---(i)

$\therefore$ The auxilliary equation, $m^2 + 1 = 0$

$$\Rightarrow m = \pm i$$

$\therefore$ The solution $y = (c_1\cos x + c_2\sin x)$ ---(ii)

By the given condition,

$c_1 = 2$ and $c_2 = -2$

$\therefore$ ~~The~~ from (ii) the solution is, $y = 2\cos x - 2\sin x$

$= 2(\cos x - \sin x)$ (Ans)

5) Solve the diff. equation $\{(D^4-1)^3(D^2-1)^2(D^2+D+1)^2\}y = 0$

$\Rightarrow$ The auxilliary equation,

$$(m^4-1)^3(m^2-1)^2(m^2+m+1)^2 = 0$$

$\therefore (m^4-1)^3 = 0$

$\therefore (m^2+1)(m^2-1) = 0$

$m = -1, 1, \pm i, \pm i, \pm i$

$m^2 - 1 = 0$

$m = \pm 1, \pm 1$

$m^2 + m + 1 = 0$

$m = \frac{-1 \pm \sqrt{1-4}}{2}$

$= \frac{-1 \pm i\sqrt{3}}{2}, \frac{-1 \pm i\sqrt{3}}{2}$

$\therefore$ The ~~solution~~, $y = (c_1 + c_2x + c_3x^2 + c_4x^3)e^x + (c_4 + c_5x + c_6x^2)e^{-x} + (c_7 + c_8x + c_9x^2)\cos x + (c_{10} + c_{11}x + c_{12}x^2)\sin x$

$\therefore m = 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, \pm i, \pm i, \pm i, -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}, -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$

$$\therefore y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4) e^{x} + (c_6 + c_7 x + c_8 x^2 + c_9 x^3 + c_{10} x^4) e^{-x}$$
$$+ (c_{11} + c_{12} x + c_{13} x^2) \cos x + (c_{14} + c_{15} x + c_{16} x^2) \sin x$$
$$+ e^{-\frac{1}{2}x}\left[(c_{17} + c_{18} x) \cos \frac{\sqrt{3}}{2} x + (c_{19} + c_{20} x) \sin \frac{\sqrt{3}}{2} x\right]$$

Particular integral

Consider the diff. equation, $F(D)y = f(x)$

Rule-I:- When $f(x) = e^{ax}$, $P.I. = \frac{1}{F(D)} f(x)$
$$= \frac{1}{F(D)} e^{ax}$$
$$= \frac{1}{F(a)} e^{ax}, \quad F(a) \neq 0$$

1) Solve the diff. equation $(D^2+1)y = e^{x}$

$\Rightarrow$ auxilliary equation is $m^2 + 1 = 0$
$$m = \pm i$$

$\therefore C.F. = c_1 \cos x + c_2 \sin x$

$\therefore P.I. = \frac{1}{D^2+1} e^{x} = \frac{1}{1^2+1} e^{x} = \frac{1}{2} e^{x}$

$\therefore$ The general solution, $y = C.F. + P.I. = c_1 \cos x + c_2 \sin x + \frac{1}{2} e^{x}$

[Note:- $\frac{1}{F(D)} c = \frac{c}{F(0)}$; $F(0) \neq 0$]

2) Solve the diff. equation $(D^2+D+1)y = 2$

$\Rightarrow$ The auxilliary equation,
$$m^2 + m + 1 = 0$$
$$\therefore m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$\therefore C.F. = e^{-\frac{1}{2}x}\left[c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right]$

$\therefore P.I. = \frac{1}{D^2+D+1} \cdot 2 = \frac{2}{0+0+1} = 2$

$\therefore$ The general solution, $y = C.F. + P.I.$

$\Rightarrow y = e^{-\frac{1}{2}x}\left[c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right] + 2$

Rule-2:- When $f(x) = \sin ax$ or $\cos ax$

$$P.I. = \frac{1}{F(D)}(\sin ax) = \frac{1}{\phi(D^2)}(\sin ax) \quad [F(D) = \phi(D^2) \text{ (Say)}]$$

$$= \frac{1}{\phi(-a^2)}(\sin ax), \quad \phi(-a^2) \neq 0$$

3) Solve the diff. equation $(D^2+4)y = \sin x$

$\Rightarrow$ The auxilliary equation,

$$m^2+4=0$$
$$\therefore m^2 = -4$$
$$\text{or, } m = \pm 2i$$

$$\therefore C.F. = (C_1\cos 2x + C_2\sin 2x)$$

$$\therefore P.I. = \frac{1}{F(D)}\sin x = \frac{1}{D^2+4}\sin x$$

$$= \frac{1}{-(1)^2+4}\sin x \quad [D^2 = -a^2]$$

$$= \frac{1}{3}\sin x$$

$\therefore$ General solution,

$$y = C_1\cos 2x + C_2\sin 2x + \frac{1}{3}\sin x.$$

4) Solve the diff. equation $(D^2+D+1)y = \cos 3x$.

$\Rightarrow$ The auxilliary equation,

$$m^2+m+1 = 0$$

$$\Rightarrow m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\therefore C.F. = e^{-\frac{1}{2}x}\left[C_1\cos\frac{\sqrt{3}}{2}x + C_2\sin\frac{\sqrt{3}}{2}x\right]$$

$$P.I. = \frac{1}{F(D)}\cos 3x = \frac{1}{D^2+D+1}\cos 3x$$

$$= \frac{1}{-9+D+1}\cos 3x = \frac{1}{D-8}\cos 3x$$

$$= \frac{D+8}{D^2-64}\cos 3x = \frac{D+8}{-9-64}\cos 3x = \frac{D+8}{-73}\cos 3x$$

$$= -\frac{1}{73}(D\cos 3x + 8\cos 3x)$$

$$= -\frac{1}{73}[-3\sin 3x + 8\cos 3x]$$

$\therefore$ The general solution, $y = e^{-\frac{1}{2}x}\left[C_1\cos\frac{\sqrt{3}}{2}x + C_2\sin\frac{\sqrt{3}}{2}x\right] - \frac{1}{73}[8\cos 3x - 3\sin 3x]$

5) Find the

$\Rightarrow \frac{1}{D^2 \cdot D + D^2}$

$= \frac{1+D}{1-D}$

6) **Rule:-3**

$\therefore$ P.I

6) Evaluate

$\Rightarrow$ P.I.

=

7) Find the

$\Rightarrow$ P.I. =

=

=

alternative

Some Results

$(1-x)^{-n} =$

$(1-x)^{-1}$

$(1+x)^{-1} =$

$(1-x)^{-2}$

$(1+x)^{-2}$

$D \equiv \frac{d}{dx}$, $\frac{1}{D} \equiv \int (\ ) dx$

$\left[F(D) = \phi(D^2) \text{ (say)}\right]$

$\neq 0$

5) Find the value of $\frac{1}{D^3+D^2+2}(\sin x)$

$\Rightarrow \frac{1}{D^2 \cdot D + D^2 + 2}\sin x = \frac{1}{-D-1+2}\sin x = \frac{1}{1-D}\sin x$

$= \frac{1+D}{1-D^2}\sin x = \frac{1+D}{1-(-1)^2}\sin x = \frac{(1+D)\sin x}{2} = \frac{1}{2}(\sin x + \cos x)$

6) **Rule :- 3 :-** $f(x) = e^{ax} V$ [V = any function of x]

$\therefore$ P.I. $= \frac{1}{F(D)} f(x) = \frac{1}{F(D)}(e^{ax} \cdot V) = e^{ax}\frac{1}{F(D+a)}(V)$

6) Evaluate $\frac{1}{D^2-1}(e^x \sin 3x)$

$\Rightarrow$ P.I. $= e^x \frac{1}{(D+1)^2-1}\sin 3x = e^x \frac{1}{D^2+2D}\sin 3x$

$= e^x \frac{1}{2D-9}\sin 3x = e^x \frac{2D+9}{4D^2-81}\sin 3x$

$= e^x \frac{2D+9}{-81-36}\sin 3x = \frac{e^x}{-117}(6\cos 3x + 9\sin 3x)$

7) Find the value of $\frac{1}{D^2-1}e^x$

$\Rightarrow$ P.I. $= \frac{1}{D^2-1}e^x = \frac{1}{D^2-1}(e^x \cdot 1) = e^x \frac{1}{(D+1)^2-1}(1)$

$= e^x \frac{1}{D^2+2D}(1) = e^x \frac{1}{D(D+2)}(1) = e^x\left[\frac{1}{D}\left\{\frac{1}{D+2}(1)\right\}\right]$

$= e^x\left[\frac{1}{D}\left(\frac{1}{0+2}\right)\right] = e^x\left[\frac{1}{D}\cdot\frac{1}{2}\right] = e^x \cdot \frac{\frac{1}{2}}{D} = \frac{e^x \cdot x}{2}$

alternative method :- $\frac{1}{D^2-1}e^x$

$= \frac{x}{2D}(e^x) = \frac{x}{2} \cdot \frac{1}{D}(e^x) = \frac{xe^x}{2}$

Some Results :-

$(1-x)^{-n} = 1 + {}^nC_1 x + {}^{n+1}C_2 x^2 + {}^{n+2}C_3 x^3 + {}^{n+3}C_4 x^4 + \cdots$ [n = integers]

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

$\cos 3x$

$= \frac{D+3}{-73}\cos 3x$

$-\frac{1}{73}[8\cos 3x - 3\sin 3x$

Rule 4 :- $f(x)$ = polynomial in x

8) Find the value of $\frac{1}{D+1}(x^3+2x+1)$

$\Rightarrow$ $\therefore$ P.I $= \frac{1}{D+1}(x^3+2x+1)$

$= (1+D)^{-1}(x^3+2x+1)$

$= (1-D+D^2-D^3+D^4-\cdots)(x^3+2x+1)$

$= (x^3+2x+1)-(3x^2+2)+(6x)-6$

$= x^3-3x^2+8x-7$

9) Evaluate $\frac{1}{(D+2)(D+1)}(x^2)$

$\Rightarrow$ $= \frac{1}{D+2}\left[\frac{1}{D+1}(x^2)\right]$

$= \frac{1}{D+2}\left[(1+D)^{-1}(x^2)\right] = \frac{1}{D+2}(1-D+D^2-D^3+\cdots)(x^2)$

$= \frac{1}{D+2}(x^2-2x+2)$

$= \frac{1}{2(\frac{D}{2}+1)}(x^2-2x+2) = \frac{1}{2}(1+\frac{D}{2})^{-1}(x^2-2x+2)$

$= \frac{1}{2}(1-\frac{D}{2}+\frac{D^2}{4}-\frac{D^3}{8}+\cdots)(x^2-2x+2)$

$= \frac{1}{2}\left\{x^2-2x+2-\frac{1}{2}(2x-2)+\frac{1}{4}(2)\right\}$

$= \frac{1}{2}\left\{x^2-2x+2-x+1+\frac{1}{2}\right\} = \frac{1}{2}(x^2-3x+\frac{7}{2})$

Rule - 5 :- If $f(x) = x^n V$ then $\frac{1}{F(D)}(x^n V) = \left(x-\frac{1}{F'(D)}F(D)\right)^n \frac{1}{F(D)}$
[V any function of x]

10) Solve the diff. equation $(D^2+1)y = x\sin 2x$

$\Rightarrow$ given equation, $(D^2+1)y = x\sin 2x$ —— (i)

The auxilliary equation of the equation (i),

$m^2+1=0$

or, $m=\pm i$

$\therefore$ C.F. $= c_1\cos x + c_2\sin x$

$\therefore P.I. = \frac{1}{(D^2+1)} x \sin 2x$

$= \left[x - \frac{1}{2D} \cdot (D^2+1)\right] \frac{1}{D^2+1} (\sin 2x)$

$= \left(x - \frac{D^2+1}{2D}\right) \frac{1}{-4+1} \sin 2x$

$= -\frac{1}{3}\left(x \sin 2x - \frac{D^2+1}{2D} \sin 2x\right)$

$= -\frac{1}{3}\left(x \sin 2x - \frac{D^2+1}{2} \times \frac{\cos 2x}{-2}\right) = -\frac{1}{3}\left(x \sin 2x + \frac{D^2+1}{4} \cos 2x\right)$

$= -\frac{1}{3}\left(x \sin 2x + \frac{1}{4} D^2 \cos 2x + \frac{1}{4} \cos 2x\right)$

$= -\frac{1}{3}\left(x \sin 2x - \frac{1}{4} \times 4 \cos 2x + \frac{1}{4} \cos 2x\right)$

$= -\frac{1}{3}\left(x \sin 2x - \cos 2x + \frac{1}{4} \cos 2x\right)$

$= -\frac{1}{3}\left(x \sin 2x - \frac{3}{4} \cos 2x\right)$

$\therefore$ The general solution, $y = C.F. + P.I. = C_1 \cos x + C_2 \sin x - \frac{1}{3}\left(x \sin 2x - \frac{3}{4} \cos x\right)$

$= C_1 \cos x + C_2 \sin x - \frac{1}{3} x \sin 2x + \frac{1}{4} \cos 2x$

11) Solve the diff. equation $(D^2+4)y = \cos 2x$

$\Rightarrow$ The given equation, $(D^2+4)y = \cos 2x$ ——(i)

The auxilliary equation, $m^2+4=0$

or, $m = \pm 2i$

$\therefore C.F. = C_1 \cos 2x + C_2 \sin 2x$

$\therefore P.I. = \frac{1}{D^2+4} \cos 2x = \frac{x}{2D} \cos 2x = \frac{x}{2} \frac{\sin 2x}{2}$

$= \frac{x \sin 2x}{4}$

$\therefore$ The general solution, $y = C.F. + P.I.$

$= C_1 \cos 2x + C_2 \sin 2x + \frac{x \sin 2x}{4}$

12) Solve the diff. equation $(D^2 - 2D + 2)y = e^x \sin 2x$

$\Rightarrow$ The given equation, $(D^2 - 2D + 2)y = e^x \sin 2x$ ——(i)

$\therefore$ the auxilliary equation $m^2 - 2m + 2 = 0$

$\therefore m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$\therefore C.F. = e^x (C_1 \cos x + C_2 \sin x)$

$\therefore P.I. = \frac{1}{D^2-2D+2} e^x \sin 2x = e^x \frac{1}{(D+1)^2 - 2(D+1) + 2} \sin 2x$

$= e^x \frac{1}{D^2+2D+1-2D-2+2} \sin 2x = e^x \frac{1}{D^2+1} \sin 2x$

$= e^x \frac{1}{-4+1} \sin 2x = \frac{e^x \sin 2x}{-3}$

$\therefore$ The general solution, $y = C.F. + P.I.$ 14)

$= e^x \{c_1 \cos x + c_2 \sin x\} - \frac{1}{3} e^x \sin 2x$.

13) Solve the diff. equation $\frac{d^2y}{dx^2} - y = x \sinh x$.

$\Rightarrow$ The given equation, $\frac{d^2y}{dx^2} - y = x \sinh x$ —(i)

the equation can be written as,

$(D^2-1)y = x \sinh x$ —(i)

the auxilliary equation,

$m^2 - 1 = 0$

$m = \pm 1$

$\therefore C.F. = c_1 e^x + c_2 e^{-x}$

$\therefore P.I. = \frac{1}{D^2-1} x \sinh x = \frac{1}{D^2-1}\left(x \cdot \frac{e^x - e^{-x}}{2}\right)$

$= \frac{1}{2} \frac{1}{D^2-1}(xe^x) - \frac{1}{2} \frac{1}{D^2-1}(xe^{-x})$

$= \frac{e^x}{2} \frac{1}{(D+1)^2-1}(x) - \frac{e^{-x}}{2} \frac{1}{(D-1)^2-1}(x)$

$= \frac{e^x}{2} \frac{1}{D(D+2)}(x) - \frac{e^{-x}}{2} \frac{1}{D(D-2)}(x)$

$= \frac{e^x}{2} \frac{1}{D}\left[\frac{1}{D+2}(x)\right] - \frac{e^{-x}}{2} \frac{1}{D}\left[\frac{1}{D-2}(x)\right]$ 15)

$= \frac{e^x}{4} \cdot \frac{1}{D}\left(1+\frac{D}{2}\right)^{-1}(x) + \frac{e^{-x}}{4} \cdot \frac{1}{D}\left(1-\frac{D}{2}\right)^{-1}(x)$

$= \frac{e^x}{4} \cdot \frac{1}{D}\left(1 - \frac{D}{2} + \frac{D^2}{4} - \cdots\right)(x) + \frac{e^{-x}}{4} \cdot \frac{1}{D}\left(1 + \frac{D}{2} + \frac{D^2}{4} + \cdots\right)(x)$

$= \frac{e^x}{4} \cdot \frac{1}{D}\left(x - \frac{1}{2}\right) + \frac{e^{-x}}{4} \cdot \frac{1}{D}\left(x + \frac{1}{2}\right)$

$= \frac{e^x}{8}\left(x - \frac{1}{2}\right)^2 + \frac{e^{-x}}{8}\left(x + \frac{1}{2}\right)^2 = \frac{e^x}{8}\left(x^2 - x + \frac{1}{4}\right) + \frac{e^{-x}}{8}\left(x^2 + x + \frac{1}{4}\right)$

$= \boxed{\frac{x^2}{4}\frac{e^x + e^{-x}}{2} - \frac{x}{4}\frac{e^x - e^{-x}}{2} + \frac{1}{16}\frac{e^x + e^{-x}}{2}} - \frac{x^2}{4}\cosh x - \frac{x}{4}\sinh x + \frac{1}{16}\cosh x$

Note

$\therefore$ The general solution, $y = C.F. + P.I.$

$= c_1 e^x + c_2 e^{-x} + \frac{e^x}{8}\left(x - \frac{1}{2}\right)^2 + \frac{e^{-x}}{8}\left(x + \frac{1}{2}\right)^2$

14) Solve the diff. equation $(D^2-2D+4)y = e^x \cos^2 x$

$\Rightarrow$ the given equation,

$$(D^2-2D+4)y = e^x \cos^2 x \quad \text{---(i)}$$

$\therefore$ the auxilliary equation,

$$m^2-2m+4=0$$

$$\Rightarrow m = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3}$$

$$\therefore C.F. = e^x[c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x]$$

$$\therefore P.I. = \frac{1}{D^2-2D+4} e^x \cos^2 x = \frac{1}{D^2-2D+4} \frac{e^x}{2}(1+\cos 2x)$$

$$= \frac{1}{2}\frac{1}{D^2-2D+4} e^x + \frac{1}{2}\frac{1}{D^2-2D+4} e^x \cos 2x$$

$$= \frac{1}{2}\frac{e^x}{1-2+4} + \frac{1}{2} e^x \frac{1}{(D+1)^2-2(D+1)+4} \cos 2x$$

$$= \frac{1}{2}\frac{e^x}{3} + \frac{1}{2} e^x \frac{1}{D^2+1+\cancel{2D}-\cancel{2D}-2+4} \cos 2x$$

$$= \frac{e^x}{6} + \frac{1}{2} e^x \frac{1}{D^2+3} \cos 2x$$

$$= \frac{e^x}{6} + \frac{1}{2} e^x \frac{1}{-4+3} \cos 2x$$

$$= \frac{e^x}{6} - \frac{e^x}{2} \cos 2x$$

$\therefore$ The general solution, $y = e^x[c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x] + \frac{e^x}{6} - \frac{e^x}{2}\cos 2x$. (Ans)

15) Solve the diff. equation $\frac{d^2y}{dx^2} - y = 1$; given that $y=0$ when $x=0$, and $y \to$ a finite limit when $x \to -\infty$

$\Rightarrow$ the given equation can be written as,

$$(D^2-1)y = 1 \quad \text{---(i)}$$

$\therefore$ auxilliary equation, $m^2-1=0$

$$m = \pm 1$$

$$\therefore C.F. = c_1 e^x + c_2 e^{-x}$$

$$\therefore P.I. = \frac{1}{D^2-1} \cdot 1 = \frac{1}{-1} = -1$$

$\therefore$ The general solution, $y = C.F. + P.I. = c_1 e^x + c_2 e^{-x} - 1$ ---(ii)

By the first given condition, $c_1 + c_2 - 1 = 0$

$$c_1 + c_2 = 1 \quad \text{---(iii)}$$

again by the 2nd condition, $c_2 = 0$

$\therefore$ from (iii), $c_1 = 1$

$\therefore y = e^x - 1$ (Ans)

16) Solve the diff. equation $(D-1)^2(D^2+1)^2 y = e^x + x$

=> Given that, $(D-1)^2(D^2+1)^2 y = e^x + x$ —(i)

∴ the auxilliary equation,

$$(m-1)^2(m^2+1)^2 = 0$$

$$\Rightarrow (m-1)^2 = 0 \quad , \quad (m^2+1)^2 = 0$$

$$m = 1, 1 \qquad m^2 = -1$$

$$m = \pm i, \pm i$$

∴ C.F. $= (c_1 + c_2 x)e^x + (c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x$

∴ P.I. $= \frac{1}{(D-1)^2(D^2+1)^2} e^x + x$

$$= \frac{1}{(D-1)^2(D^2+1)^2} e^x + \cancel{\frac{1}{(D-1)^2(D^2+1)^2} x}$$

$$= \frac{x}{2(D-1)(D^2+1)^2 + 2(D^2+1)2D(D-1)^2} e^x + \dots$$

$$= \frac{x}{2(D^2+1)(D-1)[D^2+1+2D(D-1)]} e^x + \dots$$

$$= \frac{x}{2(D^2+1)(D-1)(3D^2+1-2D)} e^x + x$$

$$= \frac{x^2}{2(D^2+1)(3D^2+1-2D) + 4D(D-1)(3D^2+1-2D) + 2(D^2+1)(D-1)(6D-2)} e^x$$

$$= \frac{x^2 e^x}{2(1+1)(3+1-2)} + x$$

$$= \frac{x^2 e^x}{4 \times 2} \cancel{+x} = \frac{x^2 e^x}{8}$$

∴ The general solution,

∴ $y =$ C.F. + P.I.

$= (c_1 + c_2 x)e^x + (c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x + \frac{x^2 e^x}{8}$

$$\frac{1}{(D-1)^2(D^2+1)^2}x$$

$$= \frac{1}{(D^2+1)^2}\cdot\frac{1}{(1-D)^2}x = \frac{1}{(D^2+1)^2}(1-D)^{-2}(x)$$

$$= \frac{1}{(D^2+1)^2}(1+2D+3D^2+\dots)(x)$$

$$= \frac{1}{(D^2+1)^2}(x+2)$$

$$= (1+D^2)^{-2}(x+2) = (1-2D^2+3D^4-\dots)(x+2)$$

$$= x+2-0 = x+2$$

$$\therefore P.I. = \frac{x^2e^x}{8} + x+2$$

$$\therefore y = C.F. + P.I. = (C_1+C_2x)e^x + (C_3+C_4x)\cos x + (C_5+C_6x)\sin x + \frac{x^2e^x}{8} + x+2.$$

Find the value of $\frac{1}{D+1}(e^{e^x})$ [There is no parameter in Particular integral]

17) $\Rightarrow$ Let, $u = \frac{1}{D+1}(e^{e^x})$

$\therefore (1+D)u = e^{e^x}$

or, $\frac{du}{dx} + u = e^{e^x}$ ——(i) [Linear in u]

$\therefore I.F. = e^{\int dx} = e^x$

$\therefore$ multiplying both sides of (i) by I.F. and integrating we have,

$u\cdot e^x = \int e^{e^x}\cdot e^x + c = \int e^z dz + c$ Let, $e^x = z$, $e^x dx = dz$

$= e^z + c = e^{e^x} + c$

or, $\therefore u = \frac{e^{e^x}}{e^x}$

General rule for P.I. :- $\boxed{\frac{1}{D+m}f(x) = e^{-mx}\int e^{mx}f(x)\,dx}$

$\therefore \frac{1}{D-m}f(x) = e^{mx}\int e^{-mx}f(x)\,dx$

18) Solve the following diff equation $\frac{d^2y}{dx^2} + a^2 y = \sec ax$.

$\Rightarrow$ the equation can be written as

$$(D^2 + a^2) y = \sec ax \text{ --- (i)}$$

the auxilliary equation,

$$m^2 + a^2 = 0$$

$$\text{or, } m = \pm ia$$

$$\therefore C.F. = c_1 \cos ax + c_2 \sin ax.$$

$$\therefore P.I. = \frac{1}{D^2 + a^2} (\sec ax)$$

$$= \frac{1}{(D - ia)(D + ia)} (\sec ax)$$

$$= \frac{1}{2ia} \left[\frac{1}{D - ia} - \frac{1}{D + ia} \right] (\sec ax)$$

$$= \frac{1}{2ia} \frac{1}{D - ia} (\sec ax) - \frac{1}{2ia} \frac{1}{D + ia} (\sec ax) \text{ --- (ii)}$$

$$= \frac{1}{2ia} e^{iax} \int e^{-iax} \sec ax \, dx - \frac{1}{2ia} e^{-iax} \int e^{iax} (\sec ax) \, dx$$

$$= \frac{1}{2ia} e^{iax} \int (\cos ax - i \sin ax) \sec ax \, dx - \frac{1}{2ia} e^{-iax} \int (\cos ax + i \sin ax) \sec ax \, dx$$

$$= \frac{1}{2ia} e^{iax} \int (1 - i \tan ax) \, dx - \frac{1}{2ia} e^{-iax} \int (1 + i \tan ax) \, dx$$

$$= \frac{1}{2ia} e^{iax} \left[x - \frac{i \log |\sec ax|}{a} \right] - \frac{1}{2ia} e^{-iax} \left[x + \frac{i \log |\sec ax|}{a} \right]$$

$\therefore$ General solution, $y = C.F. + P.I.$

$$= c_1 \cos ax + c_2 \sin ax + \frac{e^{iax}}{2ia} \left[x - \frac{i \log |\sec ax|}{a} \right] - \frac{e^{-iax}}{2ia} \left[x + \frac{i \log |\sec ax|}{a} \right]$$

Higher order exact diff. equation

Consider the diff. equation $P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = R$, where $P_0, P_1, P_2, \dots, P_n$ and R are functions of x.

the diff. equation is exact if.

$$P_n - P'_{n-1} + P''_{n-2} - \dots (-1)^n P_0^{(n)} = 0$$

1) Solve the diff. equation $(2x^2+3x)\frac{d^2y}{dx^2} + (6x+3)\frac{dy}{dx} + 2y = (x+1)e^x$

⇒ given equation,

$$(2x^2+3x)\frac{d^2y}{dx^2} + (6x+3)\frac{dy}{dx} + 2y = (x+1)e^x \quad \text{(i)}$$

Here, $P_0 = 2x^2+3x$, $P_1 = 6x+3$ and $P_2 = 2$

we have,

$$P_2 - P'_1 + P''_0$$

$$= 2 - 6 + 4 = 0$$

∴ The equation (i) is exact.

we have,

$$(2x^2+3x)y_2 + (6x+3)y_1 + 2y$$

$$\frac{d}{dx}\{(2x^2+3x)y_1\} = (2x^2+3x)y_2 + (4x+3)y_1$$

$$2xy_1 + 2y$$

$$\frac{d}{dx}(2xy) = 2xy_1 + 2y$$

$$0$$

∴ The first integral is $(2x^2+3x)y_1 + 2xy = \int(x+1)e^x dx + c_1$

$$= \int xe^x dx + \int e^x dx + c_1$$

$$= xe^x - e^x + e^x + c_1$$

$$= xe^x + c_1$$

$$\therefore (2x^2+3x)\frac{dy}{dx} + 2xy = xe^x + c_1$$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{2x+3}y = \frac{xe^x + c_1}{2x^2+3x} \quad \text{(ii)} \quad \text{[Linear in 'y']}$$

$$\therefore \text{I.F.} = e^{\int \frac{2dx}{2x+3}} = e^{\log(2x+3)} = 2x+3$$

multiplying both sides of (ii) by I.F. and integrating we have,

$$y(2x+3) = \int \frac{xe^x + c_1}{2x^2+3x}(2x+3)\,dx + c_2$$

$$= \int \frac{xe^x + c_1}{x}\,dx + c_2$$

$$= \int \left(e^x + \frac{c_1}{x}\right)dx + c_2$$

$$= e^x + c_1 \log x + c_2$$

$$\therefore y = \frac{e^x + c_1 \log x + c_2}{2x+3}$$

2) Solve the diff. equation $(1+x+x^2)\frac{d^3y}{dx^3} + (3+6x)\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 0$

=> given equation is,

$$(1+x+x^2)\frac{d^3y}{dx^3} + (3+6x)\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 0 \quad \text{—— (i)}$$

Here, $P_0 = 1+x+x^2$, $P_1 = 3+6x$, $P_2 = 6$ & $P_3 = 0$

We have

$$P_3 - P_2' + P_1'' - P_0'''$$

$$= 0 - 0 + 0 - 0 = 0$$

$\therefore$ Equation (i) is exact.

$$(1+x+x^2)y_3 + (3+6x)y_2 + 6y_1 = 0$$

$$\frac{d}{dx}\{(1+x+x^2)y_2\} = (1+x+x^2)y_3 + (1+2x)y_2$$

$$(2+4x)y_2 + 6y_1$$

$$\frac{d}{dx}\{(2+4x)y_1\} = (2+4x)y_2 + 4y_1$$

$$2y_1$$

$$\frac{d}{dx}(2y) = 2y_1$$

$$0$$

$\therefore$ First integral is,

$$(1+x+x^2)\frac{d^2y}{dx^2} + (2+4x)\frac{dy}{dx} + 2y = c_1 \quad \text{—— (ii)}$$

For the equation (ii),

$P_0 = 1+x+x^2, \quad P_1 = 2+4x$ and $P_2 = 2$

$\therefore$ We have,

$P_2 - P_1' + P_0''$

$= 2 - 4 + 2$

$= 0$

$\therefore$ Equation (ii) is exact again.

$\therefore \quad (1+x+x^2)y_2 + (2+4x)y_1 + 2y$

$\frac{d}{dx}\{(1+x+x^2)y_1\} = (1+x+x^2)y_2 + (1+2x)y_1$

$(1+2x)y_1 + 2y$

$\frac{d}{dx}\{(1+2x)y\} = (1+2x)y_1 + 2y$

0

$\therefore$ Second integral is,

$$(1+x+x^2)\frac{dy}{dx} + (1+2x)y = c_1x + c_2 \quad \text{---(iii)}$$

from (iii),

$$\frac{dy}{dx} + \frac{1+2x}{1+x+x^2}y = \frac{c_1x+c_2}{1+x+x^2} \quad \text{---(iv)} \quad [\text{Linear in } y]$$

$\therefore$ I.F. $= e^{\int \frac{1+2x}{1+x+x^2}dx} = 1+x+x^2$

$\therefore$ multiplying both sides of (iv) by I.F. and integrating we have,

$$y(1+x+x^2) = \int (c_1x+c_2)\,dx + c_3$$

$$= \frac{c_1x^2}{2} + c_2x + c_3$$

$$\therefore y = \frac{\frac{c_1x^2}{2} + c_2x + c_3}{1+x+x^2} = \frac{c_1x^2 + 2c_2x + 2c_3}{2(1+x+x^2)} \quad \text{(Ans)}$$

Alternative, the first integral of (i),

$$P_0\frac{d^2y}{dx^2} + (P_1 - P_0')\frac{dy}{dx} + (P_2 - P_1' + P_0'')y = \int 0\,dx + c_1$$

$$\Rightarrow (1+x+x^2)\frac{d^2y}{dx^2} + (3+6x-1-2x)\frac{dy}{dx} + (6-6+2)y = c_1$$

$$\Rightarrow (1+x+x^2)\frac{d^2y}{dx^2} + (2+4x)\frac{dy}{dx} + 2y = c_1$$

Again,

the diff. Equation (ii) is exact.

$\therefore$ Second integral is,

$$P_0 \frac{dy}{dx} + (P_1 - P_0')\, y = \int c_1\, dx + c_2$$

$$\Rightarrow (1+x+x^2)\frac{dy}{dx} + (1+2x)\, y = c_1 x + c_2 \quad \text{--- (iii)}$$

Clairaut's Equation

the diff. Equation of the form $y = px + f(P)$ — (i), where $P = \frac{dy}{dx}$ is Known as clairaut's Equation.

diff. both sides of (i) w. r. to x,

$$P = P + x\frac{dP}{dx} + f'(P)\frac{dP}{dx}$$

$$\Rightarrow \frac{dP}{dx}\left(x + f'(P)\right) = 0 \quad \text{--- (ii)}$$

from (ii), $\frac{dP}{dx} = 0$ — (iii) or $x + f'(P) = 0$ — (iv)

~~or, $x + f'$~~

from (iii), $P = c$ (constant)

$\therefore$ from (i), the general solution of the clairaut's Equation is $y = cx + f(c)$ — (v)

[Note :— the solution obtained by eleminating P between (i) and (iv) is Known as singular solution.]

1) Obtained the complete primitive and singular solution of the diff. equation $y = px + \frac{m}{p}$, ~~m being a parameter~~.

$\Rightarrow$ the given equation is, $y = px + \frac{m}{p}$ — (i)

~~diff~~ the equation (i) is in clairaut's form.

diff. both sides of (i) w. r. to x,

$$P = P + x\frac{dP}{dx} - \frac{m}{P^2}\frac{dP}{dx}$$

$$\Rightarrow \frac{dP}{dx}\left(x - \frac{m}{P^2}\right) = 0 \quad \text{--- (ii)}$$

from (ii), $\frac{dP}{dx} = 0$ — (iii) or $x - \frac{m}{P^2} = 0$ — (iv) from (iii) integrating, $p = c$ (constant)

$\therefore$ the ~~ge~~ complete primitive of (i) is, $y = cx + \frac{m}{c}$ —(v)

Singular solution :-

$\boxed{\text{Method 1}}$ [Elemination method] :—

from (iv), we have,

$$\frac{m}{p^2} = x$$

$$\Rightarrow p^2 = \frac{m}{x}$$

$$\Rightarrow p = \sqrt{\frac{m}{x}}$$

$\therefore$ from (i),

$$y = \sqrt{\frac{m}{x}} \cdot x + m\sqrt{\frac{x}{m}} = 2\sqrt{xm}$$

$$y^2 = 4mx \quad \text{—(vi)}$$

Equation (vi) is the singular solution of (i).

$\boxed{\text{Method - 2}}$ (P - discriminant method) :- the given diff. equation (i) can be written as,

$$py = p^2x + m$$

$$\Rightarrow xp^2 - yp + m = 0 \quad \text{—(vii)}$$

Equation (vii) is quadratic in P.

$\therefore$ The singular solution is obtained by equating the discriminant of (vii) to zero, which gives,

$$(-y)^2 - 4mx = 0$$

$$y^2 = 4mx$$

$\boxed{\text{Method - 3}}$ (C - discriminant method) :- the equation (v) can be written as,

$$c^2x - yc + m = 0 \quad \text{—(viii)}$$

equation (viii) is quadratic in 'c'.

$\therefore$ The singular solution is obtained by equating the discriminant of (viii) to zero, which gives,

$$y^2 = 4mx$$

$\boxed{\text{Method - 4}}$:- diff. both sides of (i) partially w. r. to P we have,

$$x - \frac{m}{p^2} = 0 \quad \text{—(ix)}$$

Eleminating p between (i) and (ix), Singular sol^n can be obtained.

From (ix), $p = \sqrt{\frac{m}{x}}$

$\therefore$ from (i), $\boxed{y^2 = 4mx}$

[Note :- The general solution (v) represents a family of straight lines. Clearly, every member of (v) touches the parabola $y^2 = 4mx$. Equivalently the parabola $y^2 = 4mx$ touches every member of the family (v). Therefore, $y^2 = 4mx$ is the envelope of the family (v). Thus, the singular solution is the ~~Em~~ Envelope of the family of curves represented by the complete primitive. This is why, singular solution is also known as envelope solution.]

2) Obtained the complete primitive and singular solution of the diff. equation $y = px + \sqrt{1+p^2}$.

$\Rightarrow$ The given equation,

$$y = px + \sqrt{1+p^2} \quad \text{—— (i)}$$

$\therefore$ (i) is in clairauts form.

$\therefore$ It's complete primitive is,

$$y = cx + \sqrt{1+c^2} \quad \text{—— (ii)},\ c \text{ being parameter.}$$

Equation (i) can be written as,

$$(y - px)^2 = 1 + p^2$$

$$\Rightarrow (x^2 - 1)p^2 - 2xy\,p + (y^2 - 1) = 0 \quad \text{—— (iii)}$$

$\therefore$ (iii) is quadratic in p.

$\therefore$ The singular solution is obtained by equating the discriminant of (iii) to zero which gives,

$$x^2y^2 - (x^2 - 1)(y^2 - 1) = 0$$

$$x^2y^2 - x^2y^2 + x^2 + y^2 - 1 = 0$$

$$\Rightarrow x^2 + y^2 = 1 \quad \text{—— (iv)}$$

$\therefore$ (iv) is the singular solution of (i).

3) Obtained the complete primitive and singular solution of the diff. equation $\sin px \cdot \cos y = \cos px \sin y + p$, $p = \frac{dy}{dx}$

$\Rightarrow$ The equation can be written as,

$$\sin(px - y) = p$$

$$\Rightarrow px - y = \sin^{-1} p$$

$$\Rightarrow y = px - \sin^{-1} p \quad \text{—— (i)}$$

$\therefore$ (i) is in clairauts form.

∴ It's complete primitive is,

$y = cx - \sin^{-1}c$ —(ii)

∴ diff. both sides of (ii) partially w.r. to p we have,

$0 = xp - \frac{1}{\sqrt{1-p^2}}$

$\Rightarrow p = \frac{1}{\sqrt{1-p^2}}$

$\Rightarrow p^2 = \frac{1}{1-p^2}$

$\Rightarrow p^2(1-p^2) = 1$

$\therefore p^2 = 1, \quad p^2 = 0$

$\therefore p = \pm 1, \quad p = 0$

∴ from (ii) when $p \neq 0$,

$y \neq 0$ —(iii)

and when $p = 1$

$y = x - \frac{\pi}{2}$ —(iv)

~~∴ Equation (iii) and (iv) are singular solution of (i).~~

and when $p = -1$,

$y = -x + \frac{\pi}{2}$ —(v)

∴ Equation (iii), (iv) and (v) are singular solution of (i).

$0 = x - \frac{1}{\sqrt{1-p^2}}$

$x^2 = \frac{1}{1-p^2}$

$1-p^2 = \frac{1}{x^2}$

$p^2 = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$

$p = \pm\frac{\sqrt{x^2-1}}{x}$

∴ from (i), when $p = +\frac{\sqrt{x^2-1}}{x}$

$y = \sqrt{x^2-1} - \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$

and when $p = -\frac{\sqrt{x^2-1}}{x}$

$\therefore y = -\sqrt{x^2-1} + \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$

4) By substitution $x^2 = u$, $y^2 = v$ reduce the equation $x^2+y^2-(p+p^{-1})xy = c^2$ to clairaut's form and find the general and singular solution.

$\Rightarrow$ ∵ $x^2 = u$ and $y^2 = v$

$\therefore 2x\,dx = du$ $\quad \therefore 2y\,dy = dv$

$\therefore q = \frac{dv}{du} = \frac{y}{x}\cdot\frac{dy}{dx} = \frac{y}{x}\cdot p$

$\therefore p = \frac{2x}{y}$

∴ The given diff equation,

$$x^2+y^2\cancel{q}-(p+p^{-1})xy=c^2 \quad \text{—(i)}$$

∴ from (i),

$$x^2+y^2-\left(\frac{qx}{y}+\frac{y}{qx}\right)xy=c^2$$

$$\Rightarrow x^2+y^2-\frac{q^2x^2+y^2}{q}=c^2$$

$$\Rightarrow x^2q+y^2q-q^2x^2-y^2=c^2q$$

$$\Rightarrow uq+vq-q^2u-v=c^2q$$

$$\Rightarrow (q-1)v=q^2u-qu+c^2q = q(q-1)u+c^2q$$

$$\Rightarrow v=qu+\frac{c^2q}{q-1} \quad \text{—(ii)}$$

Equation (ii) is in clairout's form.

∴ It's general solution is,

$$v=c'u+\frac{c^2c'}{c'-1}$$

$$y^2=c'x^2+\frac{c^2c'}{c'-1}$$

the equation (ii) can be written as,

$$v(q-1)=q(q-1)u+c^2q$$

$$\Rightarrow q^2u+(c^2-u-v)q+v=0 \quad \text{—(iii)}$$

Equation (iii) is quadratic in q.

∴ Singular sol^n is obtained by equating the discriminant of (iii) to zero,

$$(c^2-u-v)^2-4uv=0$$

$$\Rightarrow (c^2-x^2-y^2)^2=4x^2y^2$$

∴ This is the required singular solution.

5) Solve the following diff. equation $(px+y)^2=py^2$ $(y=u,\ v=xy)$

⇒ given that, $(px+y)^2=py^2$ —(i)

~~The gir~~ given that, $y=u$ and $v=xy$

∴ $dy=du$ ∴ $dv=y\,dx+x\,dy$

$$\therefore q=\frac{dv}{du}=\frac{y\,dx+x\,dy}{dy}$$

$$=x+y\frac{dx}{dy}=x+\frac{y}{p}$$

$$\therefore \frac{y}{p}=q-x$$

$$\therefore p=\frac{y}{q-x}$$

$\therefore$ from (i),

$$\left(\frac{qy}{q-x}+y\right)^2 = \frac{y}{q-x}\,q^2$$

$$\Rightarrow \left(\frac{x+q-x}{q-x}\right)^2 = \frac{y}{q-x}$$

$$\Rightarrow q^2 = y(q-x) = qy - xy = qu - v$$

$$\Rightarrow v = qu - q^2 \quad \text{(iii)}$$

$\therefore$ It is in clairaut's form.

$\therefore$ The general solution of (iii),

$$v = cu - c^2 \quad \text{(iv)}$$

$$\Rightarrow xy = cy - c^2 \quad \text{(iv)}$$

The equation (iii) can be written as,

$$q^2 - qu + v = 0$$

$\therefore$ Singular solution of (iii),

$$u^2 - 4v = 0$$

$\therefore$ Singular solution of (i),

$$y^2 = 4xy.$$

6) Find the complete primitive and singular solution of the diff. equation $xp^2 - 2yp + x + 2y = 0$ $(x^2 = u,\ y - x = v)$

$\Rightarrow$ $\because x^2 = u$, $\quad y - x = v$

$2x\,dx = du$, $\quad dy - dx = dv$

$$\therefore q = \frac{dv}{du} = \frac{dy - dx}{2x\,dx} = \frac{1}{2x}\cdot\frac{dy}{dx} - \frac{1}{2x} = \frac{1}{2x}p - \frac{1}{2x}$$

$$= \frac{1}{2x}(p-1)$$

$\therefore p - 1 = 2xq$

$\therefore p = 2qx + 1$

given equation,

$$xp^2 - 2yp + x + 2y = 0 \quad \text{(1)}$$

$$xp^2 - 2y(2qx+1) + x + 2y = 0$$

$$\Rightarrow x(2qx+1)^2 - 2y(2qx+1) + x + 2y = 0$$

$$\Rightarrow (4x^3)q^2 + (4x^2 - 4xy)q + (x - \cancel{2y} + x + \cancel{2y}) = 0$$

$$\Rightarrow (4x^3)q^2 + (4x^2 - 4xy)q + 2x = 0$$

$$\Rightarrow (4xu)q^2 + (4x^2 - 4xy)q + 2x = 0$$

$$\Rightarrow (4xu)q^2 - 4x(y-x)q + 2x = 0$$

$$\Rightarrow (4xu)q^2 - (4xv)q + 2x = 0$$

$$\Rightarrow (4u)q^2 - (4v)q + 2 = 0 \quad (2u)q^2 - (2v)q + 1 = 0$$

$\Rightarrow 2qv = 2q^2u + 1$

$\Rightarrow v = \frac{2q^2u}{2q} + \frac{1}{2q} = qu + \frac{1}{2q}$ —(ii)

$\therefore$ It is in clairaut's form.

$\therefore$ General solution,

$$v = cu + \frac{1}{2c}$$

$$\Rightarrow y - x = cx^2 + \frac{1}{2x^2}$$

Equation (ii) can be written as,

$$2q^2u - 2vq + 1 = 0$$

$\therefore$ The singular solution,

$$4v^2 - 4.2u = 0$$

$$v^2 = 2u$$

$$\Rightarrow (y-x)^2 = 2x^2$$

$\Rightarrow y^2 - 2xy - x^2 = 0$.

Home-Work :-

1) Obtained the complete primitive and singular sol^n of the Clairaut's equation,

i) $y = px + p - p^2$, ii) $y = px + \sin^{-1}p$, iii) $y = px + \sqrt{a^2p^2 + b^2}$,

iv) $p = \log(px - y)$, v) $p^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0$

2) The following equations are deducible to clairaut's form under suitable substitution, obtain the complete premitive,

i) $x^2(y - px) = p^2y$, $(x^2 = u, y^2 = v)$

ii) $(px - y)(x - py) = 2p$ $(x^2 = u, y^2 = v)$

iii) $x^2p^2 + yp(2x+y) + y^2 = 0$ $(y = u, v = xy)$

iv) $(px^2 + y^2)(px + y) = (p+1)^2$ $(u = xy, v = x+y)$

v) $y^2(y - xp) = x^4p^2$ $(x = \frac{1}{u}, y = \frac{1}{v})$

1) i) the given equation,

$$y = px + (p - p^2) \quad \text{---(1)}$$

Equation (i) is in clairaut's form.

$\therefore$ the complete primitive, $y = cx + c - c^2$

the equation (i) can be written as,

$$p^2 - (x+1)p + y = 0$$

$\therefore$ The singular solution, $(x+1)^2 - 4y = 0$

$$(x+1)^2 = 4y$$

ii) the given equation, $y = px + \sin^{-1} p$ ---(i)

Equation (i) is in clairaut's form.

$\therefore$ The complete primitive, $y = cx + \sin^{-1} c$.

diff. both sides of (i) partially w.r. to p we have,

$$0 = x + \frac{1}{\sqrt{1-p^2}}$$

$$\Rightarrow x^2(1-p^2) = 1$$

$$\Rightarrow x^2 - x^2p^2 = 1$$

$$\Rightarrow x^2 = x^2p^2 + 1$$

$$\Rightarrow x^2 - 1 = x^2p^2$$

$$\Rightarrow p = \pm \frac{\sqrt{x^2-1}}{x}$$

When $p = \frac{\sqrt{x^2-1}}{x}$, $y = \sqrt{x^2-1} + \sin^{-1}\frac{\sqrt{x^2-1}}{x}$

When $p = -\frac{\sqrt{x^2-1}}{x}$, $y = -\sqrt{x^2-1} - \sin^{-1}\frac{\sqrt{x^2-1}}{x}$

iii) $y = px + \sqrt{a^2p^2 + b^2}$ ---(i)

Equation (i) is in clairaut's form

$\therefore$ The complete primitive, $y = cx + \sqrt{b^2 + a^2c^2}$

Equation (i) can be written as,

$$(y - px)^2 = a^2p^2 + b^2$$

$$\Rightarrow y^2 + p^2x^2 - 2xyp = a^2p^2 + b^2$$

$$\Rightarrow (x^2 - a^2)p^2 - 2xyp + (y^2 - b^2) = 0$$

$\therefore$ The Singular solution, $x^2y^2 - (x^2 - a^2)(y^2 - b^2) = 0$

$$\Rightarrow x^2y^2 - x^2y^2 + x^2b^2 + a^2y^2 - a^2b^2 = 0$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

iv) given equation, $p = \log(px - y)$

$$\Rightarrow e^p = px - y$$

$$\Rightarrow y = px - e^p \quad \text{---(i)}$$

Equation (i) is in clairaut's form.

$\therefore$ The complete primitive, $y = cx - e^c$

diff. both sides of (i) partially w.r. to P we have,

$$0 = x - e^{p}$$

$$\Rightarrow x = e^{p}$$

$$\Rightarrow p = \log x$$

∴ from (i), the singular solution, $y = x\log x - x$

v) The given equation,

$$p^2(x^2-a^2) - 2xyp + (y^2-b^2) = 0 \quad \text{—(i)}$$

$$\Rightarrow p^2x^2 - p^2a^2 - 2xyp + y^2 - b^2 = 0$$

$$\Rightarrow y(y-2xp) = b^2 + a^2p^2 - x^2p^2$$

$$\Rightarrow y^2 - 2xpy + x^2p^2 = b^2 + a^2p^2$$

$$\Rightarrow (y-xp)^2 = b^2 + a^2p^2$$

$$\Rightarrow y - xp = \sqrt{b^2+a^2p^2}$$

$$\Rightarrow y = xp + \sqrt{b^2+a^2p^2} \quad \text{—(ii)}$$

Equation (ii) is in clairout's form.

∴ The complete primitive, $y = xc + \sqrt{b^2+a^2c^2}$

∴ the singular solution, $x^2y^2 = (y^2-b^2)(x^2-a^2) = x^2y^2 - a^2y^2 - b^2x^2 + a^2b^2$

$$\Rightarrow a^2y^2 + b^2x^2 = a^2b^2$$

$$\Rightarrow \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$$

2) i) ∵ $x^2 = u$ and $y^2 = v$

$2x\,dx = du$, $2y\,dy = dv$

∴ $q = \frac{dv}{du} = \frac{y}{x}p \Rightarrow p = \frac{xq}{y}$

The given equation,

$$x^2(y - px) = p^2y$$

$$\cancel{x^2}\left(y - \frac{x^2q}{y}\right) = \frac{\cancel{x^2}q^2}{y}$$

$$y^2 - x^2q = q^2$$

$$\Rightarrow v - uq = q^2$$

$\Rightarrow v = uq + q^2$, which is in clairout's form.

∴ The complete primitive,

$$v = uc' + c'^2$$

$$y^2 = x^2c' + c'^2$$

∴ The singular solution, $u^2 = 4v$

$$\Rightarrow x^4 = 4y^2$$

ii) $\because x^2 = u$ and $y^2 = v$

$2x\,dx = du$, $2y\,dy = dv$

$\therefore q = \frac{dv}{du} = \frac{y}{x}\cdot p$

$\Rightarrow p = \frac{qx}{y}$

$\therefore$ given that, $(px-y)(x-py) = 2p$

$\Rightarrow \left(\frac{qx^2}{y} - y\right)(x - qx) = \frac{2qx}{y}$

$\Rightarrow (qx^2 - y^2)(x - qx) = 2qx$

$\Rightarrow (qx^2 - y^2)(1 - q) = 2q$

$\Rightarrow qx^2 - q^2x^2 - y^2 + qy^2 = 2q$

$\cancel{\Rightarrow q^2x^2 - (x^2+y^2)q + y^2 + 2q = 0}$

$\cancel{\Rightarrow q^2x^2 - (x^2+y^2+2)q + y^2 = 0}$

$\cancel{\Rightarrow q^2u - (u+v+2)q + v = 0}$

$\Rightarrow q^2x^2 + y^2 - qy^2 - qx^2 + 2q = 0$

$\Rightarrow x^2q^2 + (2 - y^2 - x^2)q + y^2 = 0$

$\Rightarrow uq^2 + (2 - v - u)q + v = 0$;

$\Rightarrow v = (u+v-2)q - uq^2$ (i)

(i) is in clairout's form,

$\therefore$ The complete primitive,

$v = (u+v-2)c - uc^2$

$\Rightarrow y^2 = (x^2+y^2-2)c - x^2c^2$

$\therefore$ The singular solution;

$(2-v-u)^2 - 4uv = 0$

$\Rightarrow (2 - y^2 - x^2)^2 = 4x^2y^2$

iii) $\because y = u$ and $v = xy$

$\therefore dy = du$, $dv = y\,dx + x\,dy$

$\therefore q = \frac{dv}{du} = \frac{y\,dx + x\,dy}{dy}$

$= \frac{y}{p} + x$

$\Rightarrow q - x = \frac{y}{p}$

$\Rightarrow p = \frac{y}{q-x}$

The given equation,

$x^2p^2 + yp(2x+y) + y^2 = 0$

$\Rightarrow \frac{x^2y^2}{(q-x)^2} + \frac{y^2}{q-x}(2x+y) + y^2 = 0$

$\Rightarrow x^2y^2 + y^2(q-x)(2x+y) + y^2(q-x)^2 = 0$

$\cancel{\Rightarrow v^2 + u^2(q-x)}$

$\Rightarrow x^2y^2 + y^2(2xq + qy - 2x^2 - xy) + y^2(q^2 + x^2 - 2qx) = 0$

$\Rightarrow x^2y^2 + 2xy^2q + qy^3 - 2x^2y^2 - xy^3 + q^2y^2 + x^2y^2 - 2qxy^2 = 0$

$\Rightarrow \cancel{v^2} + \cancel{2qvu} + qu^3 - \cancel{2v^2} - vu^2 + q^2u^2 + \cancel{v^2} - \cancel{2qvu} = 0$

$\Rightarrow qu^3 - vu^2 + q^2u^2 = 0$

$\Rightarrow qu - v + q^2 = 0$

$\Rightarrow v = qu + q^2$ (ii) which is in clairout's form.

$\therefore$ The complete primitive, $v = cu + c^2$

$\Rightarrow xy = cy + c^2$

$\therefore$ The singular solution, $u^2 + 4v = 0$

$\Rightarrow y^2 + 4xy = 0$

$\Rightarrow y + 4x = 0$

iv) $\because$ ~~$u = xy$ and $v = x+y$~~

~~$du = y\,dx + x\,dy \qquad dv = dx+dy$~~

~~$\therefore q = \dfrac{dv}{du} = \dfrac{dx+dy}{y\,dx + x\,dy} = \dfrac{dx}{y\,dx+x\,dy} + \dfrac{dy}{y\,dx+x\,dy}$~~

v) $\because x = \dfrac{1}{u}, \quad y = \dfrac{1}{v}$ $\qquad \therefore q = \dfrac{dv}{du} = \dfrac{x^2\,dy}{y^2\,dx}$

$\Rightarrow u = \dfrac{1}{x}, \quad v = \dfrac{1}{y}$ $\qquad = \dfrac{x^2}{y^2} p$

$du = -\dfrac{1}{x^2}dx, \quad dv = -\dfrac{1}{y^2}dy$ $\qquad \Rightarrow p = \dfrac{qy^2}{x^2}$

$\therefore y^2(y - xp) = x^4 p^2$

$\Rightarrow y^2\left(y - \dfrac{qy^2}{x}\right) = q^2 y^4$

$\Rightarrow xy - qy^2 = xy^2 q^2$

$\Rightarrow q^2 = \dfrac{1}{y} - \dfrac{q}{x} = v - qu$

$\Rightarrow v = qu + q^2$ —(i), which is in clairaut's form.

$\therefore$ The complete primitive, $v = cu + c^2$

$\Rightarrow \dfrac{1}{y} = \dfrac{c}{x} + c^2$

$\therefore$ The singular solution, $u^2 + 4v = 0$

$\Rightarrow \dfrac{1}{x^2} + \dfrac{4}{y^2} = 0$

$\Rightarrow y^2 + 4x^2 = 0$

iv) $\because u = xy$ and $v = x+y$

$du = y\,dx + x\,dy, \quad dv = dx + dy$

$\therefore q = \dfrac{dv}{du} = \dfrac{dx+dy}{y\,dx + x\,dy} = \dfrac{1 + \frac{dy}{dx}}{y + x\frac{dy}{dx}} = \dfrac{1+p}{y+xp}$

$\Rightarrow qy + qxp = 1 + p$

$\Rightarrow p(qx - 1) = 1 - qy$

$\Rightarrow p = \dfrac{1-qy}{qx-1}$

$\therefore (px^2 + y^2)(px + y) = (p+1)^2$

$\Rightarrow \left(\dfrac{x^2 - qyx^2}{qx-1} + y^2\right)\left(\dfrac{x - xqy}{qx-1} + y\right) = \left(\dfrac{1-qy}{qx-1} + 1\right)^2$

$\Rightarrow \dfrac{x^2 - qyx^2 + qxy^2 - y^2}{\cancel{qx-1}} \times \dfrac{x - xqy + qxy - y}{\cancel{qx-1}} = \dfrac{(1-qy+qx-1)^2}{\cancel{(qx-1)^2}}$

$\Rightarrow \{x^2 - y^2 + qxy(y-x)\} \times (x-y) = q^2(x-y)^2$

$\Rightarrow (x+y)(x-y) - qxy(x-y)^{\cancel{2}} = q^2(x-y)^{\cancel{2}}$

$\Rightarrow v - qu(x-y) = q^2(x-y)$

$\Rightarrow v -$

$\Rightarrow \{(x+y) - qxy\} = q^2$

$\Rightarrow v - qu = q^2$

$\Rightarrow v = qu + q^2$ ——(i) which is in clairaut's from

$\therefore$ The complete primitive, $V = cu + c^2$

$\Rightarrow x+y = cxy + c^2$

$\therefore$ The singular solution,

$u^2 + 4v = 0$

$x^2y^2 + 4(x+y) = 0$

Rules for determining I.F.

Homogenious function :- A function $f(x,y)$ is said to be homogenious in x,y of degree 'n' if $f(tx, ty) = t^n f(x,y)$ or, $f(x,y) = x^n \varphi\left(\frac{y}{x}\right)$ or $f(x,y) = y^n \psi\left(\frac{x}{y}\right)$.

For example, the function $f(x,y) = ax^2 + 2hxy + by^2$ is homogenious in x,y of degree '2' since $f(tx, ty) = a(tx)^2 + 2h(tx)(ty) + b(ty)^2$
$= t^2(ax^2 + 2hxy + by^2) = t^2 f(x,y)$

The diff. equation $Mdx + Ndy = 0$ is said to be homogenious if ~~as~~ M and N are both homogenious in x,y of same degree.

Rule -I :-
i) If the equation $Mdx + Ndy = 0$ is both homogenious and exact (degree of homogeniating $\neq -1$), then $Mx + Ny = c$ is the solution of the equation.
ii) If the equation $Mdx + Ndy = 0$ is ~~not~~ be homogenious but not exact and $Mx + Ny \neq 0$, then it's I.F. $= \frac{1}{Mx + Ny}$

1) Solve the diff. equation $3x^2y\,dx + (x^3 + y^3)\,dy = 0$
$3x^2y\,dx + (x^3+y^3)dy = 0$ ——(i)
$\Rightarrow$ Here, $M = 3x^2y$ and $N = x^3 + y^3$
clearly, M and N are both homogenious in x,y of same degree 3.
$\therefore$ The equation (i) is homogenious.
Again, $\frac{\partial M}{\partial y} = 3x^2$ and $\frac{\partial N}{\partial x} = 3x^2$
$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation (i) is exact.
Thus, the equation (i) is homogenious and exact.
$\therefore$ It's solution is $Mx + Ny = c$ [Rule-I (i)]
$\Rightarrow 3x^3y + x^3y + y^4 = c$
$\Rightarrow 4x^3y + y^4 = c$

2) Solve the diff. equation $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$
$(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy$ ——(i)
$\Rightarrow$ Here, $M = x^2y - 2xy^2$, $N = -x^3 + 3x^2y$
clearly, M and N are both homogenious in x,y of same degree 3.
$\therefore$ The equation (i) is homogenious.

Now, $\frac{\partial M}{\partial y} = x^2 - 4xy \neq \frac{\partial N}{\partial x} = -3x^2 + 6xy$

$\therefore$ (i) is not exact.

$\therefore$ Thus, the equation (i) is homogenious but not exact.

We have,

$$Mx + Ny = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2 \neq 0$$

$\therefore$ By Rule I(ii), I.F. $= \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$

Multiplying both sides of (i) by I.F. we have,

$$\frac{x^2y - 2xy^2}{x^2y^2}dx - \frac{x^3 - 3x^2y}{x^2y^2}dy = 0$$

$$\Rightarrow \left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \quad \text{—— (ii)}$$

$$\Rightarrow -\frac{2}{x}dx + \frac{3}{y}dy + \frac{dx}{y} - \frac{x\,dy}{y^2} = 0$$

$$\Rightarrow -\frac{2}{x}dx + \frac{3}{y}dy + \frac{y\,dx - x\,dy}{y^2} = 0$$

$$\Rightarrow -\frac{2}{x}dx + \frac{3}{y}dy + d\left(\frac{x}{y}\right) = 0$$

$\therefore$ Integrating we have,

$$-2\log x + 3\log y + \frac{x}{y} = C$$

[Note:- The equation (ii) is exact.

We have, $\int M dx = \frac{x}{y} - 2\log x$ and $\int N dy = \frac{x}{y} + 3\log y$

$\therefore$ The general solution, $\frac{x}{y} + 3\log y - 2\log x = C$]

Rule-II:- If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ = function of x alone $= f(x)$ (say)

Then I.F. of the equation $Mdx + Ndy = 0$ is $e^{\int f(x)dx}$.

3) Solve the diff. equation $(x^3 + xy^4)dx + 2y^3 dy = 0$

$\Rightarrow$ The given equation,

$$(x^3 + xy^4)dx + 2y^3 dy = 0 \quad \text{—— (i)}$$

Here, $M = x^3 + xy^4$, $N = 2y^3$

We have, $\frac{\partial M}{\partial y} = 4xy^3 \neq \frac{\partial N}{\partial x} = 0$, (i) is not exact.

We have, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4xy^3}{2y^3} = 2x$ = function of x alone.

$\therefore$ By Rule II, I.F. $= e^{\int 2x\,dx} = e^{x^2}$

Multiplying both sides of (i) by I.F. we have,

$(x^3 e^{x^2} + xy^4 e^{x^2})\,dx + 2e^{x^2} y^3\,dy = 0$

$\Rightarrow x^3 e^{x^2}\,dx + xy^4 e^{x^2}\,dx + 2e^{x^2} y^3\,dy = 0$

$\Rightarrow x^3 e^{x^2}\,dx + \frac{1}{2} d(e^{x^2} y^4) = 0$

Integrating we have,

$\int x^3 e^{x^2}\,dx + \frac{1}{2} e^{x^2} y^4 = c$

$\Rightarrow \int z e^z\,dz + \frac{1}{2} e^{x^2} y^4 = c$

$\Rightarrow ze^z - e^z + \frac{1}{2} e^{x^2} y^4 = c$

$\Rightarrow 2x^2 e^{x^2} - 2e^{x^2} + e^{x^2} y^4 = c$

Rule - III :- If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ = function of y alone $= g(y)$ (say) then I.F. of the equation $M\,dx + N\,dy = 0$ is $e^{-\int g(y)\,dy}$.

4) Solve the diff. eq^n $(2xy^4 e^y + 2xy^3 + y)\,dx + (x^2 y^4 e^y - x^2 y^2 - 3x)\,dy = 0$

$\Rightarrow$ the given equation, $(2xy^4 e^y + 2xy^3 + y)\,dx + (x^2 y^4 e^y - x^2 y^2 - 3x)\,dy = 0$ ——(i)

Here, $M = 2xy^4 e^y + 2xy^3 + y$, $N = x^2 y^4 e^y - x^2 y^2 - 3x$

$\frac{\partial M}{\partial y} = 8xy^3 e^y + 2xy^4 e^y + 6xy^2 + 1 \neq \frac{\partial N}{\partial x} = 2xy^4 e^y - 2xy^2 - 3$

$\therefore$ Equation (i) is not exact.

$\therefore$ we have,

$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{8xy^3 e^y + 8xy^2 + 4}{2xy^4 e^y + y + 2xy^3}$

$= \frac{4(2xy^3 e^y + 2xy^2 + 1)}{y(2xy^3 e^y + 2xy^2 + 1)} = \frac{4}{y}$ = function of y alone.

$\therefore$ By Rule III, $I.F. = e^{-4\int \frac{dy}{y}} = e^{-4\log y} = y^{-4} = \frac{1}{y^4}$

$\therefore$ Multiplying both sides of (i) by I.F. we have,

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right)dy = 0$$

$\therefore \int M\,dx = x^2e^y + \frac{x^2}{y} + \frac{x}{y^3}$ (assuming y constant)

$\therefore \int N\,dy = x^2e^y + \frac{x^2}{y} + \frac{x}{y^3}$ (assuming x constant)

$\therefore$ The general solution,

$$x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$$

Rule - IV :- If $M = yf_1(xy)$ and $N = xf_2(xy)$ and $Mx - Ny \neq 0$ then the I.F. of $M\,dx + N\,dy = 0$ is $\frac{1}{Mx - Ny}$.

We have,

$$M\,dx + N\,dy = \frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

$$= \frac{1}{2}\left[(Mx+Ny)\,d(\log xy) + (Mx-Ny)\,d\left(\log \frac{x}{y}\right)\right]$$

$$\frac{M\,dx + N\,dy}{Mx - Ny} = \frac{1}{2}\left[\frac{Mx+Ny}{Mx-Ny}\,d\log(xy) + d\log\left(\frac{x}{y}\right)\right]$$

5) Solve the diff. Eq^n $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$

$\Rightarrow (x^2y^2 + xy + 1)y\,dx - (x^2y^2 - xy + 1)x\,dy = 0$ —— (i)

~~Here, $M = (x^2y^2 + xy + 1)y$~~

$\therefore Mx - Ny = x^3y^3 + \cancel{x^2y^2} + xy + x^3y^3 - \cancel{x^2y^2} + xy$

$= 2(x^3y^3 + xy) \neq 0$

$\therefore$ By Rule IV I.F. $= \frac{1}{Mx - Ny} = \frac{1}{2xy(x^2y^2+1)}$

Multiplying both sides of (i) by I.F. we have,

$$\left(\frac{1}{2x} + \frac{y}{2(x^2y^2+1)}\right)dx + ($$

∴ multiplying by I.F. we have,

$\frac{1}{2}\left[\cancel{\frac{Mx+Ny}{}}\right.$ $\frac{M\,dx+N\,dy}{Mx-Ny}=0$

$\therefore \frac{1}{2}\left[\frac{Mx+Ny}{Mx-Ny}\,d(\log xy)+d\left(\log\frac{x}{y}\right)\right]=0$

$\Rightarrow \frac{xy}{x^2y^2+1}\,d\log(xy)+d\log\left(\frac{x}{y}\right)=0$

$\Rightarrow \frac{xy}{x^2y^2+1}\,\frac{d(xy)}{xy}+d\log\frac{x}{y}=0$

$\Rightarrow \frac{d(xy)}{1+(xy)^2}+d\left(\log\frac{x}{y}\right)=0$

Integrating,

$\tan^{-1}(xy)+\log\left(\frac{x}{y}\right)=c$ (Ans)

Rule - V :- (i) If p and q are constant then $x^{p-1}y^{q-1}$ is the I.F. of the eqn $py\,dx+qx\,dy=0$

(ii) If $x^{r-1-m}\times y^{s-1-n}$ is an I.F. of the eqn $x^m y^n(ry\,dx+sx\,dy)=0$

(iii) $x^h y^k$ is an I.F. of the eqn $py\,dx+qx\,dy+x^m y^n(ry\,dx+sx\,dy)=0$

if $\frac{h+1}{p}=\frac{k+1}{q}$ and $\frac{h+m+1}{r}=\frac{k+n+1}{s}$

iv) $x^h y^k$ is an I.F. of the eqn $x^\alpha y^\beta(my\,dx+nx\,dy)+x^{\alpha_1}y^{\beta_1}(m_1y\,dx+n_1x\,dy)=0$

if $\frac{h+\alpha+1}{m}=\frac{k+\beta+1}{n}$ and $\frac{h+\alpha_1+1}{m_1}=\frac{k+\beta_1+1}{n_1}$

6) Solve the diff. eqn $(y^3-2yx^2)\,dx+(2xy^2-x^3)\,dy=0$

$\Rightarrow (y^3-2yx^2)\,dx+(2xy^2-x^3)\,dy=0$ ——(i)

∴ The equation can be written as,

$(y^3dx+2xy^2dy)+(-2yx^2dx-x^3dy)=0$

$\Rightarrow y^2(y\,dx+2x\,dy)+x^2(-2y\,dx-x\,dy)=0$ ——(ii)

$\Rightarrow x^0y^2(y\cdot 1\,dx+2\cdot x\,dy)+x^2y^0(-2\cdot y\cdot dx-1\cdot x\,dy)=0$

Here, $\alpha=0,\ \beta=2,\ m=1,\ n=2$

and $\alpha_1=2,\ \beta_1=0,\ m_1=-2,\ n_1=-1$

Let, $x^h y^k$ be an I.F. of (i).

$\therefore$ we have,

$$\frac{h+\alpha+1}{m} = \frac{k+\beta+1}{n} \quad \text{— (ii)}$$

and $$\frac{h+\alpha_1+1}{m_1} = \frac{k+\beta_1+1}{n_1} \quad \text{— (iii)}$$

from (ii),

$$\therefore \frac{h+0+1}{1} = \frac{k+2+1}{2}$$

$\Rightarrow 2h+2 = k+3$

$\Rightarrow 2h-k = 1$ — (iv)

from (iii), $$\frac{h+2+1}{+2} = \frac{k+0+1}{+1}$$

$h+3 = 2k+2$

$\Rightarrow 2k-h = 1$

$\Rightarrow h = 2k-1$ — (v)

$\therefore$ from (iv), $2(2k-1) - k = 1$

$\Rightarrow 4k - 2k - k = 1$

$\Rightarrow k = 1$

$\therefore h = 1$

$\therefore$ The I.F. $= xy$

Separable Eq^n

1) Solve the diff. Eq^n $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

$\Rightarrow$ The given equation is, $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$ ___ (i)

dividing both sides of (i) by $\sqrt{1-y^2}\sqrt{1-x^2}$ we get,

$$\frac{x\,dx}{\sqrt{1-x^2}} + \frac{y\,dy}{\sqrt{1-y^2}} = 0$$

Integrating, $\int \frac{x\,dx}{\sqrt{1-x^2}} + \int \frac{y\,dy}{\sqrt{1-y^2}} = c$

$\Rightarrow -\sqrt{1-x^2} - \sqrt{1-y^2} = c$

$\Rightarrow \sqrt{1-x^2} + \sqrt{1-y^2} + c = 0$

2) Solve the diff. Eq^n $2y\,dx - x\,dy = xy^3\,dy$

$\Rightarrow$ The given eq^n can be written as,

$2y\,dx - x(1+y^3)\,dy = 0$ ___ (i)

dividing both sides of (i) by xy we get,

$$\frac{2\,dx}{x} - \frac{1+y^3}{y}\,dy = 0$$

$\therefore$ Integrating, $2\int \frac{dx}{x} - \int \frac{dy}{y} - \int y^2\,dy = c$

$\Rightarrow 2\log x - \log y - \frac{y^3}{3} = c$

$\Rightarrow 6\log x - 3\log y - y^3 = c'$

$\Rightarrow \log \frac{x^6}{y^3} - y^3 = c'$

3) Solve the diff. Eq^n $\frac{dy}{dx} = \sin(x+y)$

$\Rightarrow$ The given Eq^n, $\frac{dy}{dx} = \sin(x+y)$ ___ (i)

Let,

$v = x+y$

$\therefore \frac{dv}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

$\therefore$ from (i), $\frac{dv}{dx} - 1 = \sin v$

$\Rightarrow \frac{dv}{dx} = 1 + \sin v$

$\Rightarrow \frac{dv}{1+\sin v} - dx = 0$

$\therefore$ Integrating,

$$\int \frac{dv}{1+\sin v} = x + c$$

$$\Rightarrow \int \frac{1-\sin v}{\cos^2 v} dv = x + c$$

$$\Rightarrow \int \sec^2 v \, dv - \int \sec v \tan v \, dv = x + c$$

$$\Rightarrow \tan v - \sec v = x + c$$

$$\Rightarrow \tan(x+y) - \sec(x+y) = x + c.$$

4) Solve the diff. eqn $\frac{dy}{dx} = \sqrt{y-x}$

$\Rightarrow$ given eqn, $\frac{dy}{dx} = \sqrt{y-x}$

Let,

$y - x = z^2$

$\therefore 2z\frac{dz}{dx} = \frac{dy}{dx} - 1$

$\Rightarrow \frac{dy}{dx} = 2z\frac{dz}{dx} + 1$

$$\Rightarrow 2z\frac{dz}{dx} + 1 = z$$

$$\Rightarrow 2z\frac{dz}{dx} = z - 1$$

$$\Rightarrow \frac{2z}{z-1} dz = dx$$

$\therefore$ Integrating,

$$2\int \frac{z}{z-1} dz = x + c$$

$$\Rightarrow 2\int \frac{z-1+1}{z-1} dz = x + c$$

$$\Rightarrow 2\int dz + 2\int \frac{dz}{z-1} = x + c$$

$$\Rightarrow 2z + 2\log|z-1| = x + c$$

$$\Rightarrow 2\sqrt{y-x} + 2\log|\sqrt{y-x} - 1| = x + c.$$

5) Solve the diff. Eq^n $\frac{dy}{dx} + 2xy = x^2+y^2$

⇒ The given Eq^n, $\frac{dy}{dx} + 2xy = x^2 + y^2$

The eq^n can be written as,

$$\frac{dy}{dx} = (y-x)^2 \quad \text{—(i)}$$

Let,

$v = y - x$

$\frac{dv}{dx} = \frac{dy}{dx} - 1$

$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} + 1$

∴ from (i),

$$\frac{dv}{dx} + 1 = v^2$$

$$\Rightarrow \frac{dv}{dx} = v^2 - 1$$

$$\Rightarrow \frac{dv}{v^2-1} = dx$$

∴ Integrating, $\int \frac{dv}{v^2-1} = x + C$

$$\Rightarrow \frac{1}{2}\log\left|\frac{v-1}{v+1}\right| = x + C.$$

6) Solve the diff. Eq^n $(x - y^2)dx + 2xy\,dy = 0$

⇒ The given Equ^n can be written as,

$$\frac{dy}{dx} = \frac{y^2 - x}{2xy} \quad \text{—(i)}$$

Let,

$y^2 = v$

$2y\frac{dy}{dx} = \frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2y}\frac{dv}{dx}$

∴ from (i), $\frac{1}{2y}\frac{dv}{dx} = \frac{v-x}{2xy}$

$$\Rightarrow \frac{dv}{dx} = \frac{v-x}{x} = \frac{v}{x} - 1$$

$$\Rightarrow \cancel{\frac{dv}{dx} + 1 = \frac{v}{x}}$$

$$\Rightarrow x\,dv = (v - x)\,dx$$

$$\Rightarrow x\,dv + (x - v)\,dx = 0$$

$$\Rightarrow x\,dx + x\,dv - v\,dx = 0$$

$$\Rightarrow \frac{x\,dx}{x^2} + \frac{x\,dv - v\,dx}{x^2} = 0$$

$$\Rightarrow \frac{dx}{x} + d\left(\frac{v}{x}\right) = 0$$

~~Let,~~

$\cancel{z = v - x}$

$\cancel{\frac{dz}{dx} = \frac{dv}{dx} - 1}$

$\cancel{\Rightarrow \frac{dz}{dx} = 1}$

$\therefore$ Integrating,

$$\log x + \frac{v}{x} = c$$

$$\Rightarrow \log x + \frac{y^2}{x} = c$$

7) Solve the diff. Eq^n $(1-x^2)dy = 2y\,dx$ and show that if $y=1$, when $x=2$ then ~~3xy+3y~~ $3xy = x+3y+1$

$\Rightarrow$ given eq^n, $(1-x^2)\,dy = 2y\,dx$ —(i)

dividing both sides of (i) we get,

$$\frac{dy}{y} = \frac{2\,dx}{(1-x^2)}$$

$\therefore$ Integrating,

$$\log y = \log\frac{1+x}{1-x} + \log c = \log\left(\frac{1+x}{1-x}c\right)$$

$$\Rightarrow y = c\,\frac{1+x}{1-x} \quad \text{—(ii)}$$

By given Condition,

$$1 = c\,{-3}$$

$$\Rightarrow c = -\frac{1}{3}$$

$\therefore$ from (ii), $y = -\frac{1}{3}\,\frac{1+x}{1-x}$

$$\Rightarrow 3y - 3xy = -1 - x$$

$$\Rightarrow 3y + 1 + x = 3xy.$$

Formation of diff. Eq^n

1) Construct a diff. Eq^n $ax^2+by^2=1$, where a and b are arbitrary constants.

$\Rightarrow$ The given Eq^n, $ax^2+by^2=1$ ——— (i)

differentiating both sides of (i) w. r. to x,

$$2ax+2by\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{ax}{by}$$

$$\Rightarrow \frac{y}{x}\frac{dy}{dx}=-\frac{a}{b} \quad \text{— (ii)}$$

differentiating again w. r. to x,

$$\frac{y}{x}\frac{d^2y}{dx^2}+\frac{dy}{dx}\,\frac{x\frac{dy}{dx}-y}{x^2}=0$$

$$\Rightarrow xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$

$$\cancel{\Rightarrow x(y+1)\frac{d^2y}{dx^2}-y\,d}$$

$\Rightarrow$

2) Form the diff. Eq^n of the following family of circles $(x-a)^2+(y-b)^2=r^2$, where a and b are parameters.

$\Rightarrow$ The given eq^n, $(x-a)^2+(y-b)^2=r^2$ ——— (i)

$\therefore$ diff. both sides of (i) w. r. to x,

$$2(x-a)+2(y-b)\frac{dy}{dx}=0 \quad \text{— (ii)}$$

diff. again w. r. to x,

$$1+(y-b)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$$

$$\Rightarrow 1+(y-b)y_2+y_1^2=0$$

$$\Rightarrow y-b=-\frac{y_1^2+1}{y_2} \quad \text{— (iii)}$$

$\therefore$ from (ii), $x-a=-(y-b)y_1$

$$=\frac{1+y_1^2}{y_2}\cdot y_1 \quad \text{— (iv)}$$

$\therefore$ from (i), $\left(\frac{1+y_1^2}{y_2}\cdot y_1\right)^2+\left(\frac{1+y_1^2}{y_2}\right)^2=r^2$

$$\Rightarrow (1+y_1^2)^2y_1^2+(1+y_1^2)^2=r^2y_2^2$$

$$\Rightarrow (1+y_1^2)^3=r^2y_2^2$$

3) Construct the diff. Eq^n from the following eq^n $y^2 = a(b^2 - x^2)$, where a and b are arbitrary constants.

$\Rightarrow$ given Eq^n, $y^2 = a(b^2 - x^2)$ —(i)

diff. w.r. to x,

$$2y\frac{dy}{dx} = -2ax$$

$$\Rightarrow \frac{y}{x}\frac{dy}{dx} = -a \quad \text{—(ii)}$$

again diff. w.r. to x,

$$\frac{y}{x}\frac{d^2y}{dx^2} + \frac{dy}{dx}\,\frac{x\frac{dy}{dx} - y}{x^2} = 0$$

$$\Rightarrow xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

4) Show that the diff. Eq^n of the family of straight line $ax + by + c = 0$ is $\frac{d^2y}{dx^2} = 0$, where a, b and c are arbitrary constants.

$\Rightarrow$ given eq^n, $ax + by + c = 0$ —(i)

diff. both sides of (i) w.r. to x,

$$a + b\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{a}{b}$$

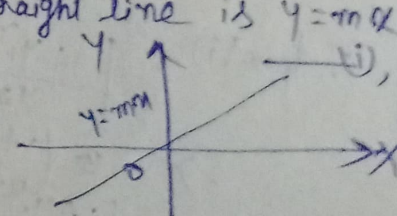
again diff. w.r. to x,

$$\frac{d^2y}{dx^2} = 0$$

[Note:- The given eq^n can be written as $Ax + By = 1$. This shows ~~one~~ ~~to~~ that the family of straight line contains atmost 2 parameters and hence the order of the diff. Eq^n of the family is 2.]

5) Find the diff. Eq^n of the family of straight lines passing through ~~to~~ origin.

$\Rightarrow$ The cartesian eq^n of the given family of straight line is $y = mx$ —(i), where m is a parameter.



$\therefore$ diff. both sides of (i) w.r. to x,

$$\frac{dy}{dx} = m \quad \text{---(ii)}$$

~~from (i)~~, eleminating the parameter m between (i) and (ii),

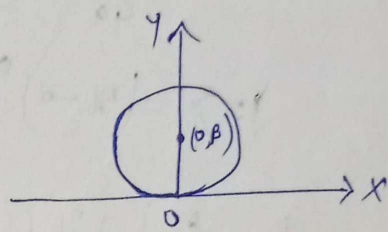
$$y = \frac{dy}{dx} x$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}.$$

$\therefore$ This is the required diff. Eq^n.

6) Obtain the diff. Eq^n of all circles each of which touches the axis of x at origin.

$\Rightarrow$ Eq^n of the given family,



$$x^2 + (y-\beta)^2 = \beta^2 \quad \text{---(i)}$$

Where, β is a parameter.

the eq^n (i) can be written as,

$$x^2 + y^2 - 2\beta y = 0 \quad \text{---(ii)}$$

from (ii),

$$\frac{x^2+y^2}{y} = 2\beta \quad \text{---(iii)}$$

diff. w.r. to x,

$$\frac{y\left(2x + 2y\frac{dy}{dx}\right) - (x^2+y^2)\frac{dy}{dx}}{y^2} = 0$$

$$\Rightarrow y\left(2x + 2y\frac{dy}{dx}\right) - (x^2+y^2)\frac{dy}{dx} = 0$$

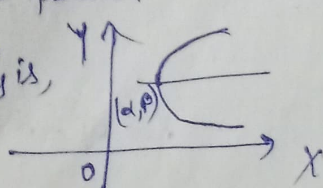
$$\Rightarrow 2xy + 2y^2\frac{dy}{dx} - x^2\frac{dy}{dx} - y^2\frac{dy}{dx} = 0$$

$$\Rightarrow (y^2 - x^2)\frac{dy}{dx} + 2xy = 0 \quad \text{---(iv)}$$

This is the required diff. Eq^n.

7) Obtain the diff. eqn for all parabola's each of which has a Latus rectum $4a$ and whose axis are parallal to the axis of x.

$\Rightarrow$ The eq^n of the given family of parabola's is,



$$(y-\beta)^2 = 4a(x-\alpha) \quad \text{---(i)}$$

Where α and β are parameters.

diff. both sides of (i) w.r. to x,

$$2(y-\beta)\frac{dy}{dx} = 4a$$

$$\Rightarrow (y-\beta)\frac{dy}{dx} = 2a$$

$$\Rightarrow y-\beta = \frac{2a}{dy/dx}$$

from (i), $4a^2 = 4a(x-\alpha)\left(\frac{dy}{dx}\right)^2$

$$\Rightarrow (x-\alpha) = \frac{a}{\left(\frac{dy}{dx}\right)^2} \quad \text{—(ii)}$$

$\therefore$ from (i),

$$\frac{4a^2}{\left(\frac{dy}{dx}\right)^2} = \underline{4a^2}$$

$$y^2+\beta^2-2y\beta = 4ax-4a\alpha$$

$$\therefore 2y\frac{dy}{dx} - 2\beta\frac{dy}{dx} = 4a$$

~~$\beta = y + \frac{y_1}{y_2}$~~

$$\Rightarrow (y-\beta)\frac{dy}{dx} = 2a$$

$$\Rightarrow (y-\beta)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow (y-\beta)y_2 + y_1^2 = 0$$

$$\Rightarrow y-\beta = -\frac{y_1^2}{y_2}$$

$\therefore$ from (i), $\frac{y_1^4}{y_2^2} = 4a(x-\alpha)$

$$\therefore x-\alpha = \frac{y_1^4}{4a y_2^2}$$

$\therefore$ from (i),

$$\cancel{\frac{y_1^{6}}{y_2^2} = \cancel{4}a \times \frac{y_1^{4}}{y_2^2 \times \cancel{4a}}}$$

$$\Rightarrow \cancel{y_1^2 = 0}$$

$$(y-\beta)^2 = 4a(x-\alpha)$$

$$\Rightarrow y^2+\beta^2-2y\beta = 4ax-4a\alpha$$

$\therefore$ diff.,

$$2yy_1-2\beta y_1 = 4a$$

$$\Rightarrow yy_1-\beta y_1 = 2a$$

$$\Rightarrow \frac{yy_1-2a}{y_1} = \beta$$

again diff.,

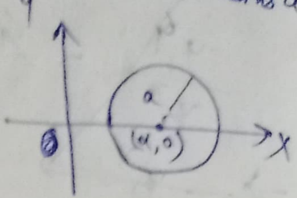
$$\frac{y_1(y_1^2+yy_2)-y_2(yy_1-2a)}{y_1^2} = 0$$

$$\Rightarrow y_1^3+\cancel{yy_1y_2}-\cancel{yy_1y_2}+2ay_2 = 0$$

$$\Rightarrow y_1^3+2ay_2 = 0$$

8) Find the diff. Eqⁿ of the system of circles of constant radius a with their centre's on the axis of x.

=> The eqⁿ of the family of circles is



$$(x-\alpha)^2 + y^2 = a^2, \ \alpha \text{ is a parameter.} \quad \text{—(i)}$$

diff. w.r. to x,

$$2(x-\alpha) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x - \alpha = -y\frac{dy}{dx} \quad \text{—(ii)}$$

∴ from (i) and (ii),

$$y^2\left(\frac{dy}{dx}\right)^2 + y^2 = a^2$$

$$\Rightarrow y^2\left[1 + \left(\frac{dy}{dx}\right)^2\right] = a^2$$

9) Find the diff. eqⁿ of all circles in XY plane.

=> The eqⁿ of the family of circles in XY plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{—(i)}$$ where f, g and c are parameters.

diff. w.r. to x,

$$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$\Rightarrow x + y\frac{dy}{dx} + g + f\frac{dy}{dx} = 0 \quad \text{—(ii)}$$

again diff.

$$\therefore \ 1 + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} + f\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \frac{1 + y_1^2 + yy_2}{y_2} = -f$$

again diff.,

$$\frac{y_2(2y_1y_2 + y_1y_2 + yy_3) - (1 + y_1^2 + yy_2)y_3}{y_2^2} = 0$$

$$\Rightarrow 3y_1y_2^2 + \cancel{yy_2y_3} - y_3 - y_1^2y_3 - \cancel{yy_2y_3} = 0$$

$$\Rightarrow 3y_1y_2^2 - y_3 - y_1^2y_3 = 0$$